

Course MMI103: homework

You can give your homework next Thursday 6th November during the course or send it by e-mail to sellier@ladhyx.polytechnique.fr once scanned (pdf file lower than 10MB) by next Friday 7th November in the evening (10 p.m).

Consider a Newtonian and homogeneous liquid, with uniform viscosity $\mu > 0$ and uniform density ρ . This liquid occupies *half* of a cylindrical solid tube with axis $(O, \underline{e}_z)$ with $\underline{e}_z$ the unit horizontal vector directed from left to right. The tube cross-section S is a disk with radius a and we shall use Cartesian coordinates (x, y, z) with origin O located on the tube axis and usual unit vectors $(\underline{e}_x, \underline{e}_y, \underline{e}_z)$. Then x and y designate the coordinates in the plane of any cross-section in which the liquid lies the lower half-disk domain $y \leq 0$ and $x^2 + y^2 \leq a$.

In each cross-section the upper half-disk domain $y \geq 0$ and $x^2 + y^2 \leq a$ is occupied by a gas which is non-viscous and has uniform density ρ_a .

The fluid and the gas are *non-miscible* and subject to a uniform gravity $\underline{g}$. This gravity has magnitude $g > 0$ and orientation α in $[0, \pi/2]$ such that $\underline{g} = g[\sin \alpha \underline{e}_z - \cos \alpha \underline{e}_y]$.

The interface between the liquid and the gas is denoted Σ and its trace in a cross-section is the segment $y = 0$ and $-a \leq x \leq a$. Due to the gravity, the liquid flows with velocity $\underline{u}$ and pressure while the gas is at rest (i.e. not flowing) with pressure p_a . The liquid flow is assumed to be steady.

1. Write the local momentum conservation law in the gas.
2. The pressure in the gas at the origin O has a given value p_0 . From the previous question then give the non-uniform pressure p_a in the gas (it is recalled that the gas density, ρ_a , is uniform).
3. The interface remains at rest. Since the fluid and the gas are non-miscible provide a condition for the liquid velocity $\underline{u}$ on the interface.
4. The interface has a given surface tension γ and unit normal $\underline{n} = \underline{e}_y$. There the liquid exerts the stress $\underline{f} = \underline{\sigma} \cdot \underline{n}$. Assuming that γ is uniform on the interface Σ provides there $\underline{f}$ in terms of p_a and $\underline{e}_y$.
5. Is the relation obtained in the previous question still valid when the interface has a non-uniform surface tension γ ? By now, we consider that γ is uniform.
6. In this problem the liquid velocity is parallel with the tube axis. Hence, one has $\underline{u} = u \underline{e}_z$. Express on the interface the stress $\underline{f}$ in terms of the liquid velocity u and pressure p .
7. From the previous questions deduce that $p = p_a$ and $\partial u / \partial y = 0$ on the interface.
8. Write the local mass conservation law in the liquid and give its consequence for the function u .
9. Calculate for the previous liquid velocity field the acceleration $d\underline{u}/dt$ where d/dt denotes the material derivative. It is recalled that the liquid flow is steady.

10. Write the local momentum conservation law in the liquid. Deduce the value of the partial derivatives $\partial p/\partial x$ and $\partial p/\partial y$. Give also the link between $\partial p/\partial z$ and Δu where Δ denotes the usual two-dimensional Laplacian for coordinates x and y .
11. From the previous question deduce the general form of the liquid pressure p .
12. Then, using the boundary condition obtained at question 7 for the pressure on the interface obtain the pressure in the liquid.
13. Deduce the value of Δu in terms of ρ, ρ_a, g, α and μ . This equation is supplemented by boundary conditions on the closed contour $\mathcal{C}$ of the cross-section. Here, $\mathcal{C}$ consists of the trace of the interface Σ and of the half-circle $y < 0$ and $x^2 + y^2 = a^2$. Give on these parts of the contour $\mathcal{C}$ the boundary conditions for u .
14. The problem obtained for u (differential equation and boundary conditions on the contour $\mathcal{C}$) is called (P) . We set $H(x, y) = x^2 + y^2 - a^2$. Calculate ΔH and examine the properties of H on the contour $\mathcal{C}$. Then deduce the liquid velocity u .
15. Give the location and the value of the maximum of u .
16. Now we consider a slipping tube. There we apply the following Navier slip boundary condition

$$\underline{u} = \frac{\lambda}{\mu} [\underline{f} - (\underline{f} \cdot \underline{n})\underline{n}], \quad \underline{f} = \underline{\sigma} \cdot \underline{n}$$

where $\underline{f}$ is the stress exerted by the flow $(\underline{u}, p)$ with stress tensor $\underline{\sigma}$, $\underline{n}$ the unit normal on the boundary *pointing into the liquid* and $\lambda \geq 0$ the tube given constant slip length.

- (a) Explain why $\underline{n} \cdot \underline{e}_z = 0$ so that we shall note $\underline{n} = n_x \underline{e}_x + n_y \underline{e}_y$ with usual unit vectors $\underline{e}_x$ and $\underline{e}_y$ parallel with the tube cross-section S .
- (b) Show that

$$\underline{f} = -p\underline{n} + \mu \left[\frac{\partial u}{\partial x} n_x + \frac{\partial u}{\partial y} n_y \right] \underline{e}_z$$

- (c) Deduce that the slip condition on the tube surface is

$$u = \lambda \left[\frac{\partial u}{\partial x} n_x + \frac{\partial u}{\partial y} n_y \right]$$

17. Express for our cylindrical tube n_x and n_y in terms of x, y and a .
18. We seek the velocity in the form $u(x, y) = A(x^2 + y^2) + B$ with unknown and constant quantities A and B . Calculate Δu and impose the boundary conditions bearing on u to get A, B and u .
19. Comment the sensitivity of u to the tube slip length λ and give the location and the value of the maximum of u for the slipping tube.
20. In a similar fashion can we obtain the velocity u of the liquid when it occupies the entire slipping cylindrical tube? If yes, give u in that case.